

Proof of validity of Riemann Hypothesis

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Mohammed Djamal CHEBBAH

Independent researcher
chebbah.djamel@gmail.com

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Abstract

This work initiates a proof of the Riemann hypothesis through a novel graphical approach based on the analytical properties of the auxiliary function $\chi(s)$. By combining symmetry constraints from the functional equation with domain exclusion principles and graphical analysis, we identify and systematically eliminate regions of the critical strip where zeros cannot occur. The application of the argument principle to the remaining domain suggests that all non-trivial zeros must lie on the critical line $\Re(s) = \frac{1}{2}$. This approach opens new perspectives for enriching the understanding of the Riemann Hypothesis.

Keywords

Riemann zeta function, χ function, non-trivial zeros, Riemann Hypothesis, zeros in the critical strip, analytic number theory.

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Contents

1	Introduction	3
1.1	Riemann Zeta Function	3
1.2	Riemann Hypothesis	3
1.3	Review of Fundamental Relations for the Zeta Function	3
1.3.1	The $\zeta(s)$ Function	3
1.3.2	The $\xi(s)$ Function	4
2	Methodology	5
2.1	Foundation	5
2.2	Domain Exclusion Principle	5
2.3	Use of Graphical Methods	5
2.4	Pedagogical Approach	6

3	Analytical Symmetries of ζ and ξ in $\mathcal{D}$	6
3.1	Symmetries of the $ \xi(s) $ Function	6
3.2	Symmetry of the $ \zeta(s) $ Function	7
3.3	Interpretation	7
3.4	Coupling of $ \zeta(s) $ Values	7
4	Study of the $\chi(s)$ Function in the Critical Strip $\mathcal{D}^+$	8
4.1	Case $\sigma = \frac{1}{2}, t > 0$	8
4.2	Graphical Study of $ \chi(s) $ Curves in the Critical Strip $\mathcal{D}^+$ for Different $t > 0$ Values	9
4.3	Study of the Derivative of $ \chi(s) $ and Introduction of the Function $g(s) = \frac{1}{ \chi(s) } \frac{d \chi(s) }{d\sigma}$	10
4.3.1	Logarithmic Derivative of $ \chi(s) $	10
4.3.2	Expression of the Derivative $\frac{d}{d\sigma} \log \chi(s) $	10
4.3.3	Definition of the Function $g(s)$	11
4.3.4	Properties of $g(s)$	11
4.4	Study of the Function $g(s)$ in the Critical Strip near $t = 6.28$	11
4.4.1	Graphical Analyses and Observations	11
4.4.2	Proof of the Extremum at $\sigma = \frac{1}{2}$	12
4.4.3	Study of the Concavity of $g(s)$	12
4.4.4	Critical Point of Negativity of $g''(s)$	13
4.4.5	Asymptotic Behavior	13
4.4.6	Study of the Sign of $g(s)$ and Characterization of $ \chi(s) $ near $t = 6.28$. Critical Values t_1, t_2, t_3	14
4.4.7	Summary	17
4.4.8	Establishment and Solution of Equations to Calculate the Exact Values of t_1, t_2 and t_3	17
5	Numerical Validation of Results	18
5.1	Proof of $g(s) > 0, 0 < t \leq t_0$	19
5.2	Verification of Concavity of $g(s)$ for $t \in [t_0, 10]$	19
5.3	Negativity of g for $t \in]t_3, 10]$	19
5.4	Asymptotic Negativity of $g(s)$ for $t \geq 10$	20
5.5	Analysis of Non-Monotonicity of $ \chi(s) $	21
5.6	Certification of Three Intersection Points of $ \chi(s) $ with the Line $ \chi(s) = 1$	21
6	Interpretation of Results	22
6.1	Validity of the Riemann Hypothesis for $0 \leq \sigma \leq 1$ and $(0 < t < t_2 \text{ or } t_3 \leq t < \infty)$	22
6.2	Absence of Zeros of ζ in the Domain $\{0 \leq \sigma \leq 1, t_2 \leq t < t_3\}$	22
6.3	Absence of Zeros on the Segment $0 < \sigma < 1, t = 0$	24
6.4	Absence of Zeros on the Vertical Lines $\sigma = 0$ and $\sigma = 1$	24
7	Conclusion	24
8	Declaration on the Use of Artificial Intelligence	25
	References	26
	Annexes	27
	Appendix 1: Interactive $\chi(s)$ function in the critical strip, t variable from 0.1 to 200	27
	Appendix 2: Simultaneous Dynamic Evolution of $g(s)$ and $\chi(s)$	29

1 Introduction

1.1 Riemann Zeta Function

The Riemann zeta function [1, p. 1], [2, p. 9, § 1.4], [3, p. 1, § 1.1], is a complex function of the variable $s = \sigma + it$, where σ and t are real, defined for $\sigma > 1$ by the Riemann series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

Through analytic continuation [1, pp. 2-3], [3, p. 13, § 2.1], Riemann succeeded as early as 1859 in connecting this function to the prime-counting function $\pi(x)$, which gives the number of primes less than or equal to a given value x . This connection relies particularly on the knowledge of the *non-trivial zeros* of $\zeta(s)$, that is, the solutions of the equation $\zeta(s) = 0$ located in the *critical strip* $0 \leq \Re(s) \leq 1$.

1.2 Riemann Hypothesis

In his paper [1, p. 5], Riemann conjectured that all non-trivial zeros of the function $\zeta(s)$ in the critical strip lie on the *critical line* $\sigma = \frac{1}{2}$. To this day, this hypothesis has been neither refuted nor confirmed. This work initiates a proof of its validity.

1.3 Review of Fundamental Relations for the Zeta Function

1.3.1 The $\zeta(s)$ Function

Conjugation of $\zeta(s)$, [3, p. 1, § 1.1]:

$$\zeta(s) = \overline{\zeta(\bar{s})} \quad (2)$$

and

$$|\zeta(s)| = |\zeta(\bar{s})|.$$

The analytic continuation of the ζ function to $\mathbb{C} \setminus \{1\}$, introduced by Riemann [3, p. 16, Eq. 2.1.9], leads to the functional equation:

$$\zeta(s) = \chi(s) \zeta(1-s). \quad (3)$$

With the value of $\chi(s)$ [3, p. 16, Eq. 2.1.10] given by:

$$\frac{\zeta(s)}{\zeta(1-s)} = \chi(s) = \frac{\pi^{s-\frac{1}{2}} \Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \quad s \in \mathbb{C} \setminus (\{1, 3, 5, \dots\} \cup \{0, -2, -4, \dots\}). \quad (4)$$

Since our study is situated in the critical strip, excluding the singular points at $s = 0$ and $s = 1$, we introduce the following subdomains:

- $\mathcal{D} = \{s \in \mathbb{C} : 0 \leq \Re(s) \leq 1\} \setminus \{0, 1\}$: the complete critical strip excluding these 2 singular points
- $\mathcal{D}^+ = \{s \in \mathbb{C} : 0 \leq \Re(s) \leq 1, \Im(s) > 0\} \setminus \{0\}$: the upper part of the critical strip, excluding the origin

Remark. In the critical strip $\mathcal{D}$, we have the identity

$$\frac{\zeta(s)}{\zeta(1-s)} = \chi(s), \quad \text{where the quotient is well-defined.}$$

The function χ is *defined and continuous* on $\mathcal{D}$. Thus, even at the non-trivial zeros s_n of $\zeta(s)$, where the left-hand expression becomes a priori indeterminate ($0/0$), we can *characterize the ratio* $R(s) = \frac{\zeta(s)}{\zeta(1-s)}$ by the continuous value of $\chi(s_n)$, obtaining a continuous extension of R to all of $\mathcal{D}$:

$$\lim_{s \rightarrow s_n} \frac{\zeta(s)}{\zeta(1-s)} = \chi(s_n).$$

In particular, taking moduli:

$$\lim_{s \rightarrow s_n} \frac{|\zeta(s)|}{|\zeta(1-s)|} = |\chi(s_n)|.$$

Comment 1. Let $R(s) = \frac{\zeta(s)}{\zeta(1-s)}$ for $s \neq s_n$. The continuity of χ on $\mathcal{D}$ allows us to resolve the indeterminacy at any non-trivial zero s_n by setting:

$$R(s_n) := \chi(s_n).$$

The singularity of R at s_n is therefore *removable*, and R becomes continuous on $\mathcal{D}$.

Comment 2. The continuity of χ on $\mathcal{D}$ is guaranteed by its holomorphy. Indeed, in the expression

$$\chi(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \pi^{s-\frac{1}{2}},$$

the poles of the numerator (at $s = 1 + 2n$) and the poles of the denominator (at $s = -2n$) are all located outside $\mathcal{D}$. Thus, χ is the ratio of two holomorphic functions on a domain where the denominator does not vanish; it is therefore holomorphic, and a fortiori continuous, on $\mathcal{D}$.

From equation (3), $\zeta(s) = \chi(s)\zeta(1-s)$, we obtain by taking moduli the relation valid everywhere in $\mathcal{D}$:

$$|\zeta(s)| = |\chi(s)| \cdot |\zeta(1-s)|. \quad (5)$$

The conjugation of the function $\chi(s)$ follows from the conjugation of its terms:

$$\chi(s) = \overline{\chi(\bar{s})} \quad \text{and} \quad |\chi(s)| = |\chi(\bar{s})| \quad (6)$$

Given the equation $\chi(s) \cdot \chi(1-s) = 1$, [3, p. 16, Eq. 2.1.11], $\chi(s)$ is different from 0 for all s . Hence:

$$|\chi(s)| \neq 0 \quad \text{for} \quad 0 \leq \sigma \leq 1, \quad \forall t > 0 \quad (7)$$

1.3.2 The $\xi(s)$ Function

The function $\xi(s)$, [1, p. 4], [3, p. 16, Eq. 2.1.1], is related to the $\zeta(s)$ function by the following relation:

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad (8)$$

The function $\xi(s)$ is holomorphic on $\mathbb{C}$, due to the factor $(s-1)$ which compensates for the pole of the $\zeta(s)$ function at $s = 1$. The function $\xi(s)$ satisfies the functional equation [3, p. 16, Eq. 2.1.13]:

$$\xi(s) = \xi(1-s) \quad (9)$$

Lemme 1.1 *In the critical strip $\mathcal{D}$, the non-trivial zeros of $\zeta(s)$ coincide exactly with those of $\xi(s)$:*

$$\xi(s) = 0 \quad \Longleftrightarrow \quad \zeta(s) = 0 \quad (10)$$

Proof. According to equation (8), for $s \neq 0$ in the critical strip, $\xi(s) = 0$ if and only if $\zeta(s) = 0$, since the other factors of $\xi(s)$ do not vanish for $0 < \sigma < 1$. The factor $(s - 1)$ compensates for the pole of $\zeta(s)$ at $s = 1$, and the gamma function has no zeros.

Lemma 1.2 *In the critical strip $\mathcal{D}$, the non-trivial zeros of $\zeta(s)$ are symmetric with respect to the point $(\frac{1}{2}, 0)$.*

For any zero s_n of $\zeta(s)$ we have:

$$\zeta(s_n) = \zeta(1 - s_n) \quad (11)$$

Proof. Let s_n be a non-trivial zero of $\zeta(s)$ in the critical strip. By Lemma 1.1, $\xi(s_n) = 0$. By the functional equation $\xi(s) = \xi(1 - s)$, we have $\xi(1 - s_n) = 0$, hence $\zeta(1 - s_n) = 0$. Thus, both s_n and $1 - s_n$ are zeros of $\zeta(s)$.

Fundamental consequence:

There cannot exist an "isolated" zero in the critical strip that would violate this symmetry.

From equation (9), we also deduce:

$$|\xi(s)| = |\xi(1 - s)| \quad (12)$$

Conjugation:

$$\xi(\bar{s}) = \overline{\xi(s)} \quad \text{and} \quad |\xi(s)| = |\xi(\bar{s})| \quad (13)$$

Symmetry of $|\xi(s)|$ with respect to the critical line $\sigma = \frac{1}{2}$:

Combining equations (12) and (13) we obtain the chain of equalities:

$$|\xi(s)| = |\xi(1 - s)| = |\xi(\overline{1 - s})|,$$

hence:

$$|\xi(s)| = |\xi(\overline{1 - s})|. \quad (14)$$

Since the points s and $\overline{1 - s}$ are symmetric with respect to the line $\sigma = \frac{1}{2}$, the function $|\xi(s)|$ is symmetric with respect to this line.

2 Methodology

2.1 Foundation

Our approach is based on the analytical framework established by Riemann for the zeta function and its associated functions, particularly the $\chi(s)$ function. It fits within the tradition of foundational work on the distribution of zeros, without recourse to recent developments in analytic number theory.

2.2 Domain Exclusion Principle

The procedure adopts a progressive exclusion strategy: by identifying regions of the critical strip where $|\chi(s)| \neq 1$, we establish that these areas cannot contain symmetric pairs of zeros. Through successive elimination, we reduce the possible domain for zeros to the critical line.

2.3 Use of Graphical Methods

Visual analysis guides intuition by revealing the geometric properties of $|\chi(s)|$ in the critical strip. Graphical representations, generated by Python scripts with Matplotlib, serve to formulate conjectures that are then demonstrated analytically.

2.4 Pedagogical Approach

The document aims to reconcile accessibility with mathematical rigor. Certain standard developments are explicitly detailed to guarantee the completeness of the reasoning and facilitate verification by the community.

3 Analytical Symmetries of $|\zeta|$ and $|\xi|$ in $\mathcal{D}$

To make visible the symmetries of the functions $|\xi(s)|$ and $|\zeta(s)|$, we have chosen to represent the modulus of these functions using a color scale in the complex plane (see Figure 1 and Figure 2). This choice highlights the analytical symmetries, makes zeros identifiable, and allows for immediate visual reading of the properties of these functions.

3.1 Symmetries of the $|\xi(s)|$ Function

The function $|\xi(s)|$, defined on the complex plane, possesses a particularly rich symmetry structure (see Figure 1).

- Symmetry with respect to the real axis (σ -axis):

$$\xi(\bar{s}) = \overline{\xi(s)} \implies |\xi(s)| = |\xi(\bar{s})|.$$

This means that $|\xi(s)|$ is symmetric with respect to the reflection $t \mapsto -t$, hence with respect to the axis $t = 0$.

- Central symmetry with respect to the point $(\frac{1}{2}, 0)$: Using the functional equation of $\xi(s)$, see equation (9), we obtain:

$$|\xi(s)| = |\xi(1-s)|.$$

This induces a central symmetry of $|\xi(s)|$ with respect to the critical point $(\frac{1}{2}, 0)$.

- Symmetry with respect to the critical line $\sigma = \frac{1}{2}$: this symmetry was demonstrated in Subsection 1.3.2, it is due to the fact that $|\xi(s)| = |\xi(\overline{1-s})|$.

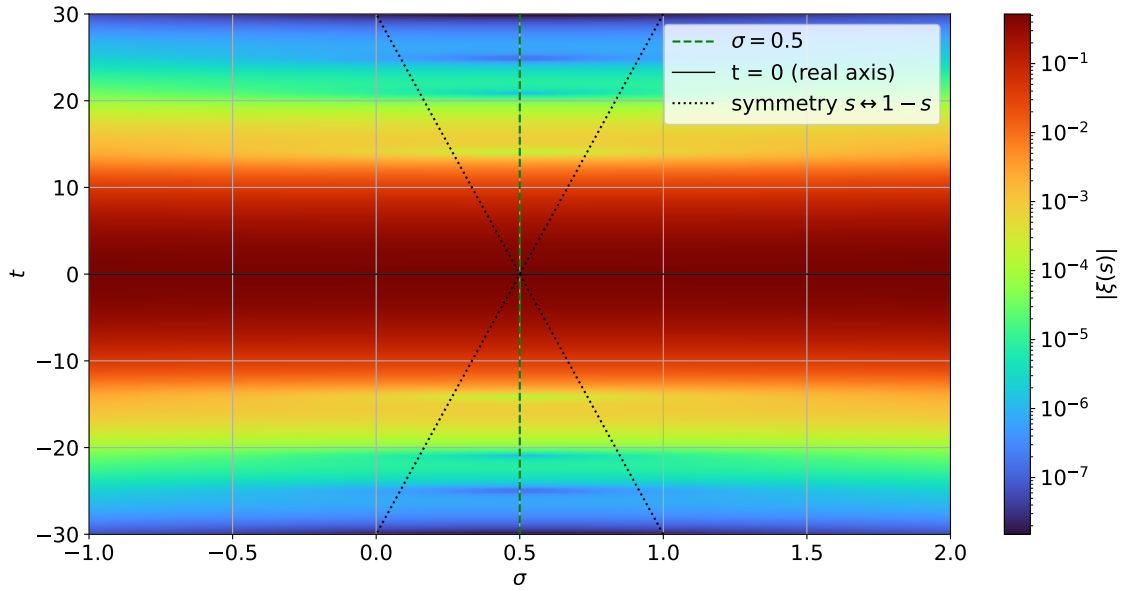


Figure 1: Symmetries of the $|\xi(s)|$ function in the complex plane

3.2 Symmetry of the $|\zeta(s)|$ Function

The function $|\zeta(s)|$, on the other hand, exhibits a more restricted symmetry structure.

- Symmetry with respect to the real axis ($t = 0$): Figure 2 shows that this unique symmetry contrasts with the triple symmetry of $|\xi(s)|$. We observe only symmetry with respect to the σ -axis, by virtue of the equality $|\zeta(s)| = |\zeta(\bar{s})|$ due to the conjugation of the $\zeta(s)$ function, see equation (2).

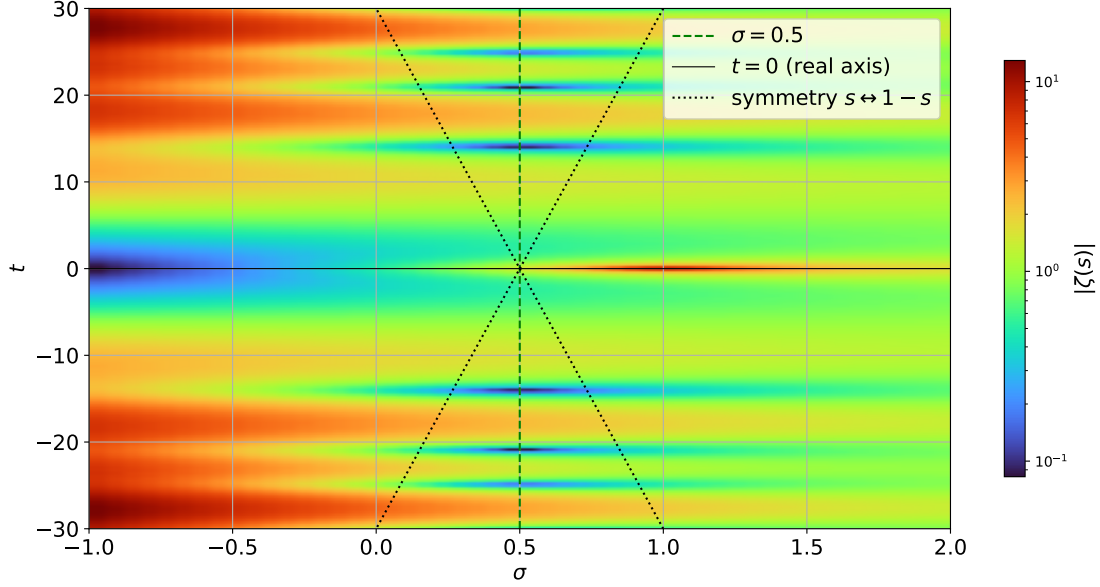


Figure 2: Symmetry of the $|\zeta(s)|$ function in the complex plane

3.3 Interpretation

The difference between the symmetry properties of the two functions suggests a strategy for approaching the Riemann Hypothesis. If we can show that the only region where symmetric pairs of zeros can appear is the line $\sigma = \frac{1}{2}$, we would obtain a proof of the hypothesis.

We will show that $|\zeta(s)|$ is asymmetric throughout the critical strip, except on the line $\sigma = \frac{1}{2}$ and for values of s in a restricted domain that we will denote D_R . In this domain, for well-determined values of s , $|\zeta(s)| \neq 0$, although $|\zeta(s)| = |\zeta(1-s)|$.

Furthermore, since the functions $|\zeta|$ and $|\chi|$, see equations (2) and (6), are symmetric with respect to the σ -axis, it suffices to limit the study to the case $t > 0$. The results can then be extended by symmetry from the positive side to the negative side of the critical strip. The segment $0 < \sigma < 1$ of the real axis and the axes $\sigma = 0$ and $\sigma = 1$, as well as the point $s = 0$, are subject to particular analysis.

3.4 Coupling of $|\zeta(s)|$ Values

From previous observations, we have noted that zeta values work in pairs, $[\zeta(s), \zeta(1-s)]$ or $[\zeta(s), \zeta(\bar{s})]$.

Lemma 1.2 establishes the symmetry of zeros of $\zeta(s)$. This symmetry also applies to the values of $|\zeta(s)|$, hence:

$$\forall s_n \text{ zero of } |\zeta(s)| \text{ in the critical strip } \mathcal{D}^+, \quad |\zeta(s_n)| = 0 \iff |\zeta(1-s_n)| = 0 \quad (15)$$

This coupling means that zeros of $|\zeta(s)|$ necessarily appear in symmetric pairs: if $|\zeta(s_n)|$ is zero, then $|\zeta(1-s_n)|$ is also zero, and conversely. In the case of moduli the symmetry is with respect to the line $\sigma = \frac{1}{2}$.

Lemma 3.1 *In the critical strip $\mathcal{D}^+$, if $|\chi(s)| = 1$ then $|\zeta(s)| = |\zeta(1-s)|$ and potential existence of zeros of $|\zeta(s)|$.*

Proof. From the functional equation $|\zeta(s)| = |\chi(s)| \cdot |\zeta(1-s)|$, equation (5):

If $|\chi(s)| = 1$, then $|\zeta(s)| = |\zeta(1-s)|$.

Two cases arise:

1. If $|\zeta(s)| \neq 0$ then $|\zeta(1-s)| \neq 0$ (situation without zeros)
2. If $|\zeta(s)| = 0$ then $|\zeta(1-s)| = 0$ by equation (15) (situation with zeros)

The condition $|\chi(s)| = 1$ is compatible with the potential existence of zeros.

Lemme 3.2 *In the critical strip $\mathcal{D}^+$, if $|\chi(s)| \neq 1$ then $|\zeta(s)| \neq |\zeta(1-s)|$, and $\zeta(s)$ has no zeros.*

Proof. From the functional equation $|\zeta(s)| = |\chi(s)| \cdot |\zeta(1-s)|$, if $|\chi(s)| \neq 1$ then necessarily $|\zeta(s)| \neq |\zeta(1-s)|$.

Now, if s were a zero of $\zeta(s)$, then by equation (15), we would also have $\zeta(1-s) = 0$, hence $|\zeta(s)| = |\zeta(1-s)| = 0$, which contradicts $|\zeta(s)| \neq |\zeta(1-s)|$. Therefore, if $|\chi(s)| \neq 1$, $\zeta(s)$ cannot have zeros.

4 Study of the $|\chi(s)|$ Function in the Critical Strip $\mathcal{D}^+$

Objective of this section. Now that we have established the crucial role of the $|\chi(s)|$ function in excluding zero-free domains, we will systematically study its behavior in the critical strip. Our goal is to precisely identify the regions where $|\chi(s)| = 1$ and the zeta function may have zeros, and the regions where $|\chi(s)| \neq 1$ in order to eliminate them from the search for zeros.

4.1 Case $\sigma = \frac{1}{2}, t > 0$

Let us begin with the simplest case: the critical line $\sigma = \frac{1}{2}$.

According to the relation $\chi(s) \cdot \chi(1-s) = 1$ [3, p. 16, Eq. 2.1.11], for $s = \frac{1}{2} + it$ we have:

$$\begin{aligned} \chi\left(\frac{1}{2} + it\right) \cdot \chi\left(1 - \left(\frac{1}{2} + it\right)\right) &= \chi\left(\frac{1}{2} + it\right) \cdot \chi\left(\frac{1}{2} - it\right) \\ &= \chi\left(\frac{1}{2} + it\right) \cdot \chi\left(\overline{\frac{1}{2} + it}\right). \end{aligned}$$

Now, as we showed previously, equation (6), we have $|\chi(s)| = |\chi(\bar{s})|$. We deduce:

$$\left| \chi\left(\frac{1}{2} + it\right) \right| \cdot \left| \chi\left(\overline{\frac{1}{2} + it}\right) \right| = \left| \chi\left(\frac{1}{2} + it\right) \right|^2 = 1.$$

$$\text{Thus, } \left| \chi\left(\frac{1}{2} + it\right) \right| = 1 \quad \forall t \in \mathbb{R}. \quad (16)$$

This equality means that on the critical line, the functional equation implies:

$$\left| \zeta\left(\frac{1}{2} + it\right) \right| = \left| \zeta\left(1 - \left(\frac{1}{2} + it\right)\right) \right| = \left| \zeta\left(\overline{1 - \left(\frac{1}{2} + it\right)}\right) \right|.$$

We observe that the points s and $\overline{1-s}$ are symmetric with respect to the line $\sigma = \frac{1}{2}$. In particular, the points $\frac{1}{2} + it$ and $1 - (\frac{1}{2} + it)$ coincide, which confirms that on the critical line, the function $|\zeta(s)|$ is symmetric, as are its zeros.

This symmetry relies precisely on the condition $|\chi(s)| = 1$. Recall that the existence of infinitely many zeros on the critical line $\sigma = \frac{1}{2}$ was demonstrated by Hardy and Littlewood as early as 1914 [3, p. 256, § 10.2].

4.2 Graphical Study of $|\chi(s)|$ Curves in the Critical Strip $\mathcal{D}^+$ for Different $t > 0$ Values

Figure 3 shows the behavior of $|\chi(s)|$ for different values of $t > 0$ in the interval $0 \leq \sigma \leq 1$. We observe that:

- All curves intersect at the critical point $p_c = \left(\frac{1}{2}, 1\right)$, graphically confirming equation (16):

$$|\chi\left(\frac{1}{2} + it\right)| = 1 \quad \forall t > 0$$

- A special case appears around $t \approx 6.28$ where the curve seems almost constant and equal to 1 over a large part of the critical strip.

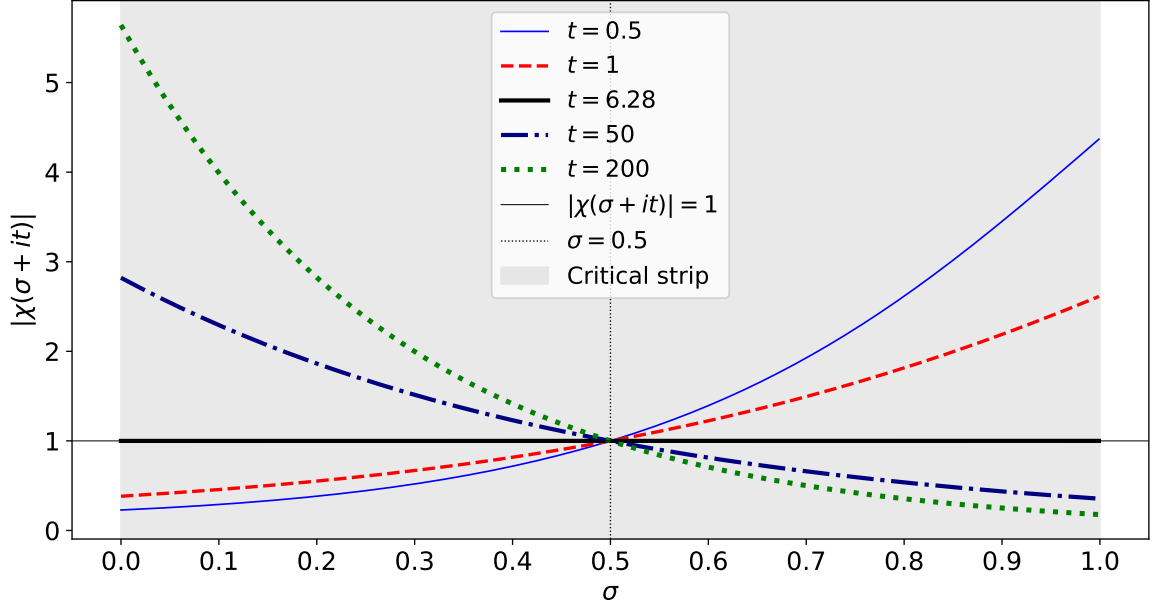


Figure 3: $|\chi(s)|$ for $0 \leq \sigma \leq 1$ and $t \in \{0.5, 1, 6.28, 50, 200\}$

Figure 4 illustrates the dynamic evolution of $|\chi(s)|$ as a function of t , revealing the progressive transition between the different observed regimes.

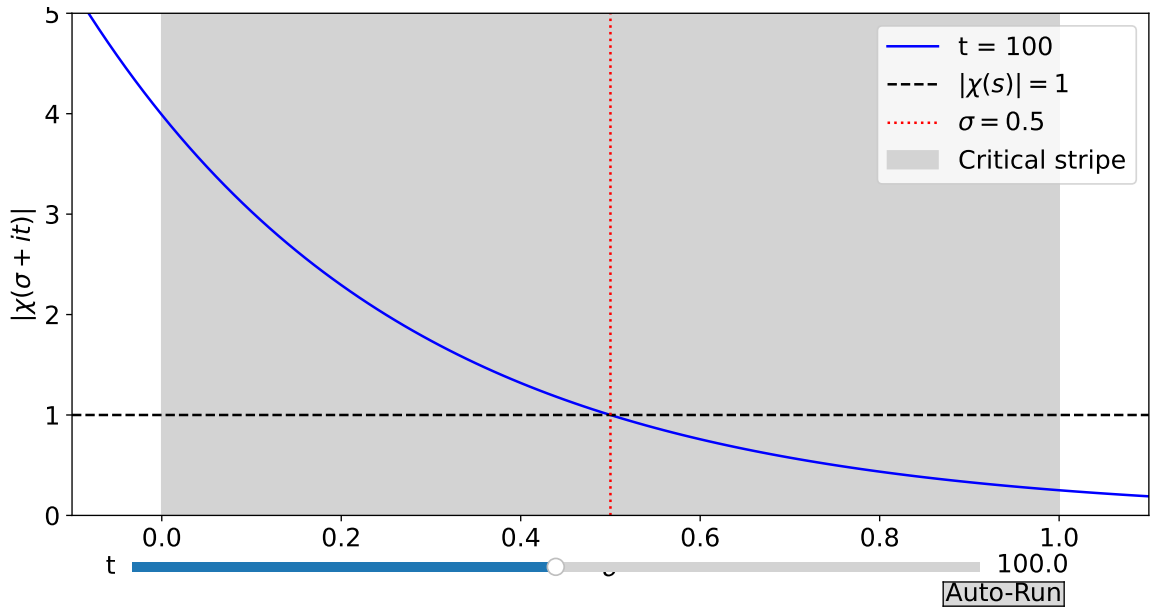


Figure 4: Dynamic evolution of $|\chi(s)|$ as a function of t in the critical strip

In the appendix, we present an interactive Python program using matplotlib that allows real-time visualization of the evolution of $|\chi(s)|$ for $0 \leq \sigma \leq 1$ (see Figure 4).

4.3 Study of the Derivative of $|\chi(s)|$ and Introduction of the Function $g(s) = \frac{1}{|\chi(s)|} \frac{d|\chi(s)|}{d\sigma}$

4.3.1 Logarithmic Derivative of $|\chi(s)|$

To precisely characterize the dynamics of $|\chi(s)|$, we examine the properties of its derivative with respect to σ . Since $|\chi(s)|$ is strictly positive, equation (7), we use its logarithmic derivative.

For a function $f(\sigma, t) \neq 0$, we have:

$$\frac{\partial}{\partial \sigma} \log(f(\sigma, t)) = \frac{\frac{\partial}{\partial \sigma} f(\sigma, t)}{f(\sigma, t)} \Leftrightarrow \frac{\partial}{\partial \sigma} f(\sigma, t) = f(\sigma, t) \cdot \frac{\partial}{\partial \sigma} \log(f(\sigma, t)) \quad (17)$$

Let $s = \sigma + it$, with σ variable and $t > 0$ a parameter. This approach allows us to differentiate with respect to σ while considering t as constant. We then have:

$$\frac{d}{d\sigma} \log(f(s)) = \frac{d}{ds} \log(f(s)) \cdot \frac{ds}{d\sigma}$$

Since $s = \sigma + it$, we have $\frac{ds}{d\sigma} = 1$, hence:

$$\frac{d}{d\sigma} \log(f(s)) = \frac{d}{ds} \log(f(s)) \quad (18)$$

Recall that if $f(s) \neq 0$:

$$f(s) = |f(s)| \cdot e^{i \arg(f(s))} \Leftrightarrow \log f(s) = \log |f(s)| + i \arg(f(s))$$

hence:

$$\log |f(s)| = \Re(\log f(s)) \quad (19)$$

Combining these results, we finally obtain:

$$\frac{d}{d\sigma} \log |\chi(s)| = |\chi(s)| \cdot \frac{d}{d\sigma} \log |\chi(s)| = |\chi(s)| \cdot \frac{d}{ds} \Re(\log \chi(s)) \quad (20)$$

Given that $|\chi(s)| > 0 \forall t$, see equation (7), we will focus only on the factor $\frac{d}{d\sigma} \log |\chi(s)|$ of the derivative, since it is this term that determines zero crossings and sign changes.

4.3.2 Expression of the Derivative $\frac{d}{d\sigma} \log |\chi(s)|$

Developing $\log \chi(s)$ from equation (4):

$$\begin{aligned} \frac{d}{ds} \log |\chi(s)| &= \frac{d}{ds} \Re \left[\log \left(\pi^{s-\frac{1}{2}} \cdot \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right) \right] \\ &= \frac{d}{ds} \Re \left[\log \pi^{s-\frac{1}{2}} + \log \Gamma\left(\frac{1-s}{2}\right) - \log \Gamma\left(\frac{s}{2}\right) \right] \\ &= \Re \left[\frac{d}{ds} \log \pi^{s-\frac{1}{2}} \right] + \Re \left[\frac{d}{ds} \log \Gamma\left(\frac{1-s}{2}\right) \right] - \Re \left[\frac{d}{ds} \log \Gamma\left(\frac{s}{2}\right) \right] \end{aligned}$$

Introducing the digamma function $\psi(s) = \frac{d}{ds} \log \Gamma(s)$ [4, p. 58, Section 6.3], we obtain:

$$\frac{d}{ds} \log |\chi(s)| = \log \pi - \frac{1}{2} \left(\Re \left[\psi\left(\frac{1-s}{2}\right) \right] + \Re \left[\psi\left(\frac{s}{2}\right) \right] \right) \quad (21)$$

4.3.3 Definition of the Function $g(s)$

We define the function:

$$g(s) = \frac{1}{|\chi(s)|} \frac{d|\chi(s)|}{d\sigma} = \log \pi - \frac{1}{2} \left(\Re \left[\psi \left(\frac{1-s}{2} \right) \right] + \Re \left[\psi \left(\frac{s}{2} \right) \right] \right) \quad (22)$$

4.3.4 Properties of $g(s)$

- **Symmetry:** $g(1-s) = g(s)$ (symmetry with respect to $\sigma = \frac{1}{2}$)
- **Constant term:** $\log \pi \approx 1.14472989$ (positive)
- **Variable term:** $-\frac{1}{2} \left(\Re \left[\psi \left(\frac{1-s}{2} \right) \right] + \Re \left[\psi \left(\frac{s}{2} \right) \right] \right)$ (can be positive or negative depending on t and σ)

The function $g(s)$ will determine the variations of $|\chi(s)|$ and notably its monotonicity.

4.4 Study of the Function $g(s)$ in the Critical Strip near $t = 6.28$

4.4.1 Graphical Analyses and Observations

The graphical analysis of $g(s)$ (Figure 5) reveals that:

- The function is symmetric with respect to the axis $\sigma = \frac{1}{2}$
- The curve presents an extremum at $\sigma = \frac{1}{2}$ for all t
- Near $t = 6.28$, the function changes sign
- For sufficiently large t , the curve becomes concave

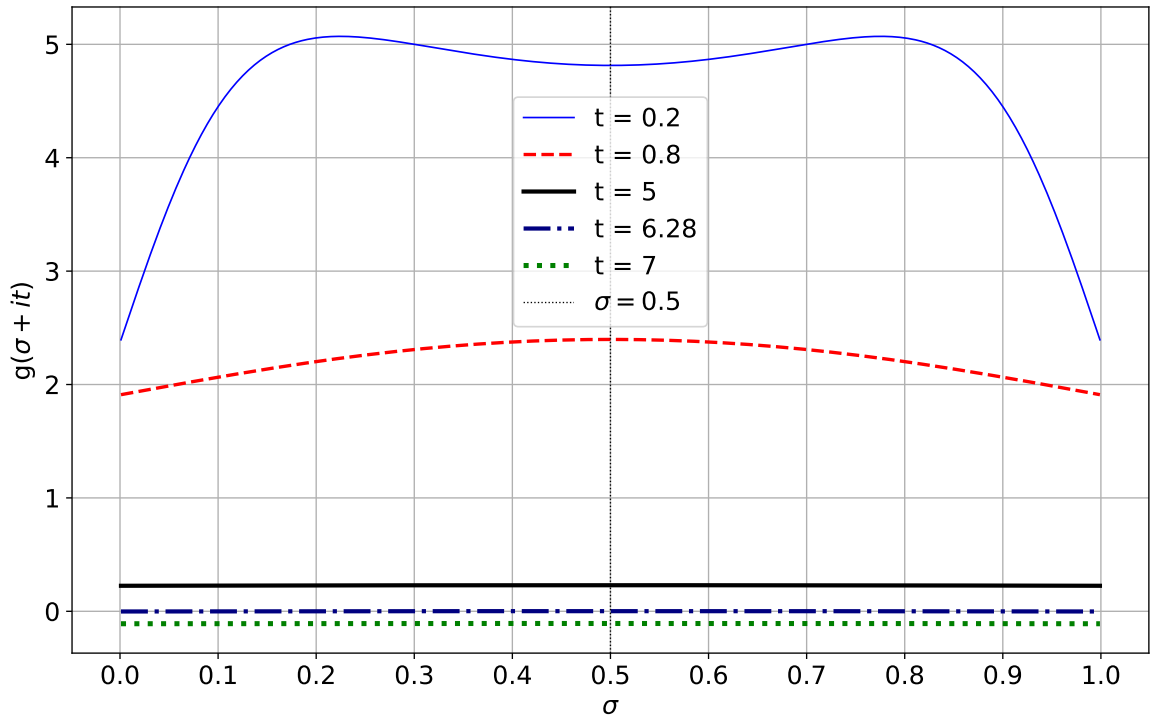


Figure 5: $g(s)$ for $s = \sigma + it$, $0 \leq \sigma \leq 1$, $t \in \{0.2, 0.8, 5, 6.28, 7\}$

4.4.2 Proof of the Extremum at $\sigma = \frac{1}{2}$

The derivative of $g(s)$ with respect to σ at $\sigma = \frac{1}{2}$ is:

$$g' \left(\frac{1}{2} + it \right) = -\frac{1}{4} \left[\Re \psi' \left(\frac{1}{4} - \frac{it}{2} \right) - \Re \psi' \left(\frac{1}{4} + \frac{it}{2} \right) \right] = 0 \quad \forall t \in \mathbb{R} \quad (23)$$

ψ' represents the polygamma function of order 1. Indeed, from $\psi'(s) = \overline{\psi'(\bar{s})}$, we deduce:

$$\psi' \left(\frac{1}{4} + \frac{it}{2} \right) = \overline{\psi' \left(\frac{1}{4} - \frac{it}{2} \right)} \quad \text{and} \quad \Re \left(\psi' \left(\frac{1}{4} + \frac{it}{2} \right) \right) = \Re \left(\psi' \left(\frac{1}{4} - \frac{it}{2} \right) \right)$$

This confirms that $g(s)$ presents an extremum at $\sigma = \frac{1}{2}$ for all t .

4.4.3 Study of the Concavity of $g(s)$

To determine the concavity of $g(s)$, we must study the sign of the second derivative of the function $g(s)$. This function is given by:

$$g''(s) = -\frac{1}{8} \left(\Re \left[\psi'' \left(\frac{1-s}{2} \right) \right] + \Re \left[\psi'' \left(\frac{s}{2} \right) \right] \right) \quad (24)$$

ψ'' represents the polygamma function of order 2. Figure 6 shows the evolution of $g''(s)$ for different values of t . We observe the symmetry with respect to $\sigma = \frac{1}{2}$ and the systematic presence of an extremum at this point for $\sigma = \frac{1}{2} \forall t$. The Figure 6 bis represents a zoom of the upper left corner of Figure 6.

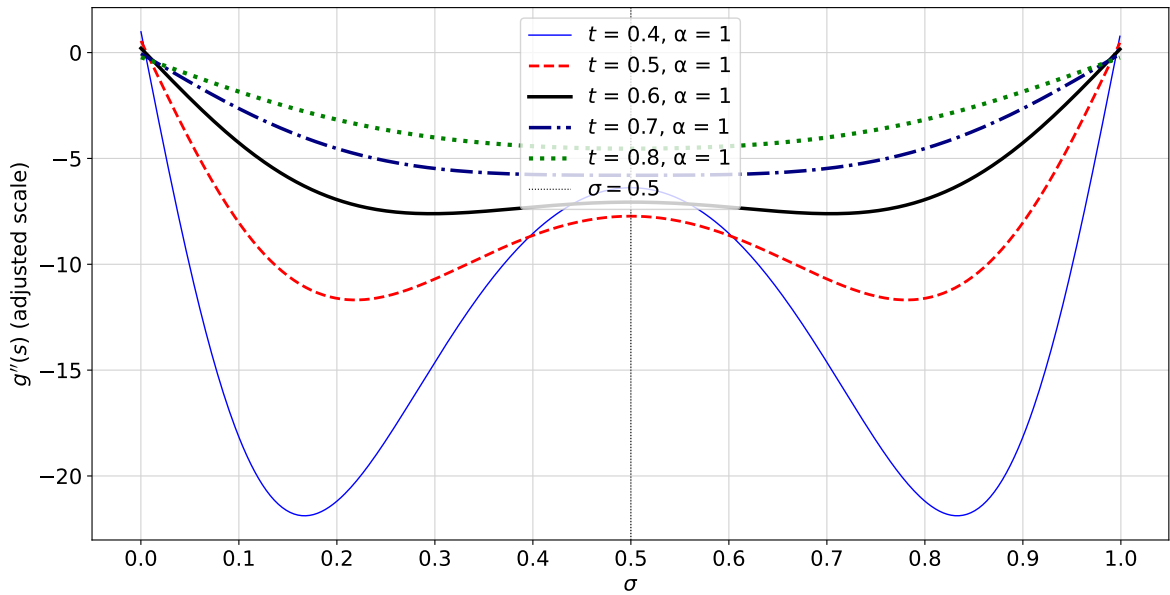


Figure 6: Evolution of $g''(s)$ for $t \in \{0.4, 0.5, 0.6, 0.7, 0.8\}$ (without scale correction α)

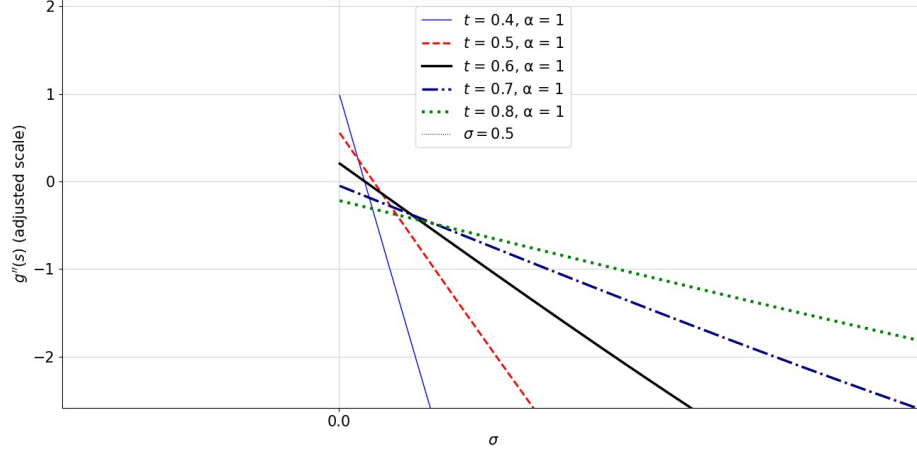


Figure 6 bis - Enlargement of Figure 6 at $\sigma = 0$ in the neighborhood of $0.4 \leq t \leq 0.8$

4.4.4 Critical Point of Negativity of $g''(s)$

After preliminary graphical research, we found that $g''(s)$ can become negative for $0.6 < t < 0.7$, see Figures 6 and 6 bis. (The multiplicative coefficient α was used during the preliminary research phase to observe the shape of the curves for different values of t). Indeed, for $0.6 < t < 0.7$, $g''(s)$ is almost convex and $g''(0, t)$ corresponds to the maximum of the curve. The entirety of $g''(s)$ is bounded by two parallel horizontal lines drawn from the points $g''(0, t)$ and $g''(0.5, t)$. The analysis of the curves $g''(0, t)$ and $g''(0.5, t)$, Figure 7, reveals that $g''(s)$ becomes strictly negative for $t > t_0 \approx 0.678157$. t_0 is the solution of the equation:

$$g''(0 + it) = 0$$

This value was determined numerically by the brentq method implemented in Python in the interval $[0.6, 0.7]$. The value $t_0 \approx 0.678156835255537$ corresponds to the point where $g''(0 + it)$ vanishes, marking the transition to strict concavity of $g(s)$ for $t > t_0$.

4.4.5 Asymptotic Behavior

For $s = it$ with $t \gg 1$, using the asymptotic expansion:

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^4} - \dots, \quad |z| \rightarrow \infty$$

we obtain:

$$g''(it) \sim -\frac{1}{t^2} + O\left(\frac{1}{t^4}\right)$$

This confirms that $g''(s)$ remains negative for large values of t .

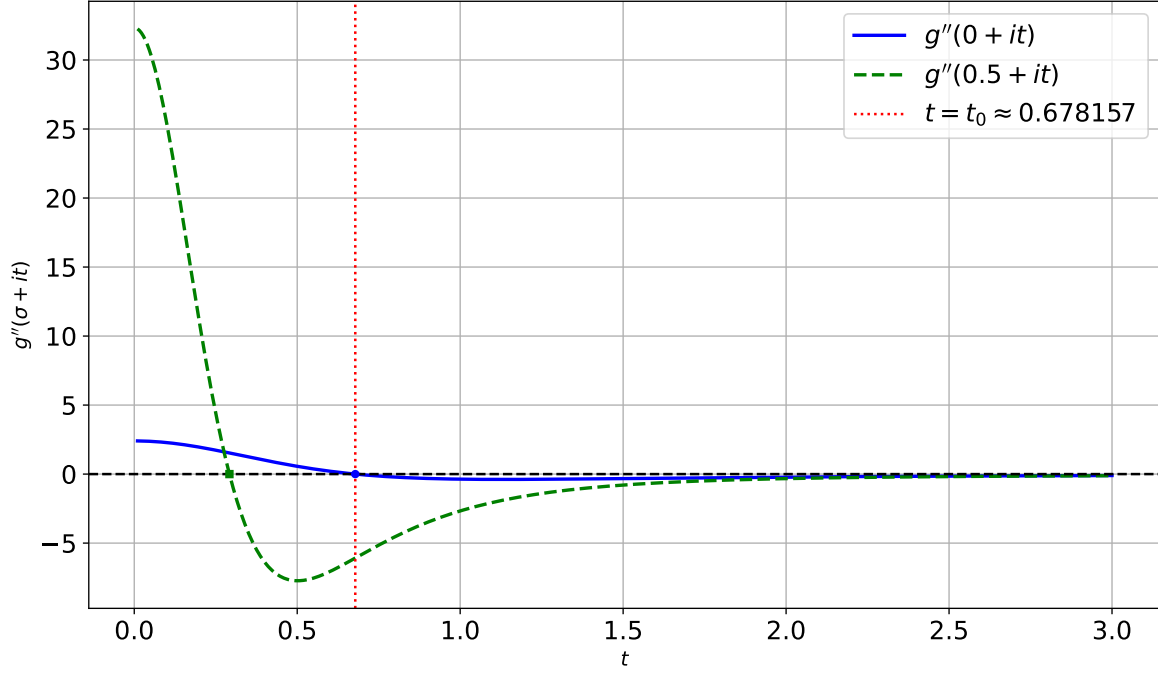


Figure 7: Critical point t_0 from which $g''(s)$ becomes strictly negative

This established concavity of $g(s)$ for $t \geq t_0$ and will allow us to rigorously determine the its sign in the vicinity of $t = 6.28$.

4.4.6 Study of the Sign of $g(s)$ and Characterization of $|\chi(s)|$ near $t = 6.28$. Critical Values t_1, t_2, t_3

Using an interactive Python program (Appendix 2), we analyzed the simultaneous behavior of $g(s)$ and $|\chi(s)|$ for different values of t in the critical strip $\mathcal{D}^+$ near $t = 6.28$. Figures 8 to 12 illustrate this study.

We found three critical values of t that allow us to describe the evolution of $g(s)$ and $\chi(s)$ simultaneously. These values define the domains where $|\chi(s)| = 1$ and where $|\chi(s)| \neq 1$. These values are:

- $t_1 \approx 6.26985$
- $t_2 \approx 6.28318$
- $t_3 \approx 6.28983$

Graphical and Analytical Study of the Properties of $|\chi(s)|$:

Regime 1: $0 < t < t_1 \approx 6.26985$, (Figure 8), This regime is decomposed into 2 phases, $t \in [0, t_0]$ and $t \in]t_0, t_1[$.

- $t \in [0, t_0]$, $g(s) > 0$ for all $\sigma \in [0, 1]$; its global minimum is positive, verified numerically, see subsection 5.1.
- $t \in]t_0, t_1[$, $g(s) > 0$ for all $\sigma \in [0, 1]$; its global minimum is positive given the concavity of $g(s)$ and the fact that $g(0, t) > 0$ for $t \leq t_1$.
- The function $|\chi(s)|$ is strictly increasing in σ .
- $|\chi(s)|$ intersects the line $|\chi(s)| = 1$ only at $\sigma = \frac{1}{2}$.

- On the critical axis $\sigma = \frac{1}{2}$, $|\zeta(s)|$ may potentially vanish.

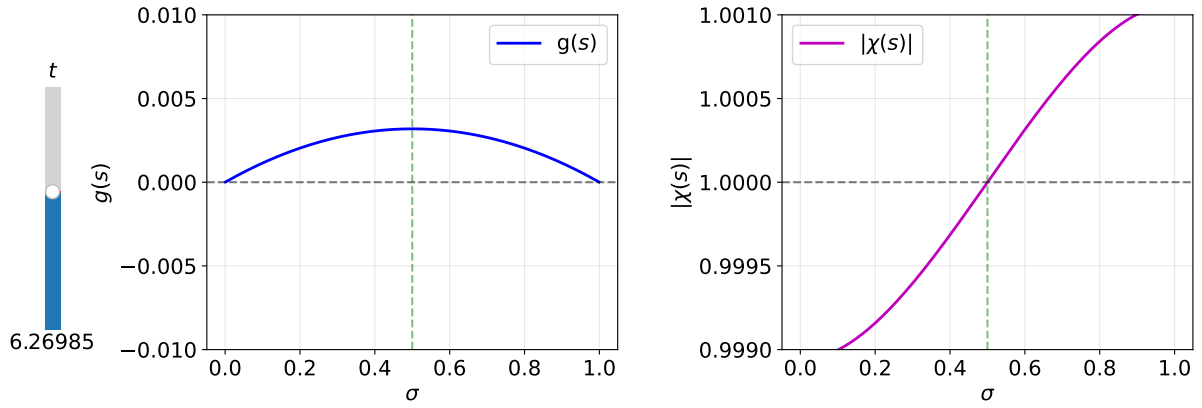


Figure 8: $g(s) > 0$ for $t < 6.26985$. $|\chi(s)|$ intersects $|\chi(s)| = 1$ only at $(\frac{1}{2}, 1)$.

Intermediate Case: t between t_1 and t_2 ($t = 6.27557$) (Figure 9)

- The sign of $g(s)$ varies with σ .
- $g(s)$ vanishes at two points symmetric with respect to $\sigma = \frac{1}{2}$.
- $|\chi(s)|$ is non-monotonic and passes through a single intersection point $(\frac{1}{2}, 1)$.
- On the critical axis $\sigma = \frac{1}{2}$, $|\zeta(s)|$ may potentially vanish.

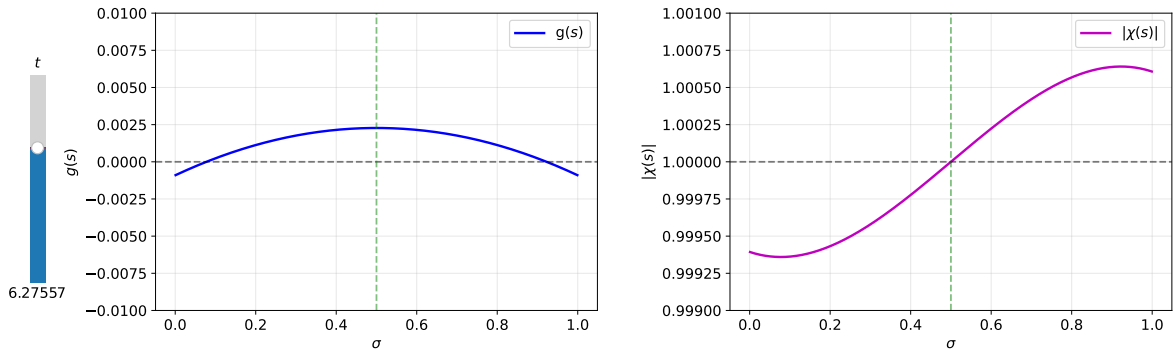


Figure 9: For $t = 6.27557$, the sign of $g(s)$ varies with σ . $|\chi(s)|$ non-monotonic intersects the line $|\chi(s)| = 1$ at a single point $(\frac{1}{2}, 1)$.

Regime 2: $t_2 \approx 6.28318$ (Figure 10)

- The sign of $g(s)$ varies with σ .
- $g(s)$ vanishes at two points symmetric with respect to $\sigma = \frac{1}{2}$.
- $|\chi(s)|$ non-monotonic passes through three intersection points centered at $(\frac{1}{2}, 1)$.
- $|\zeta(s)| = |\zeta(1-s)|$ for three values of σ centered on $\sigma = \frac{1}{2}$, giving three possibilities for $|\zeta(s)| = 0$.

- If t decreases, we return to a situation similar to the previous intermediate case (Figure 9), where $|\chi(s)|$ non-monotonic passes through one intersection point at $\left(\frac{1}{2}, 1\right)$.

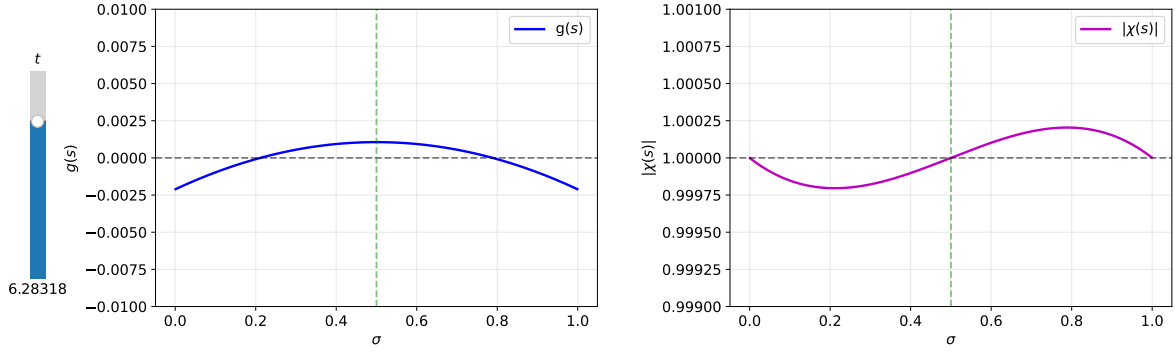


Figure 10: For $t = t_2 \approx 6.28318$, the sign of $g(s)$ varies with σ . The curve $|\chi(s)|$ intersects the line $|\chi(s)| = 1$ at a single point $\left(\frac{1}{2}, 1\right)$ and passes through three points if t increases.

Intermediate Case: t between t_2 and t_3 ($t = 6.28512$) (Figure 11)

- The sign of $g(s)$ varies with σ .
- $g(s)$ vanishes at two points symmetric with respect to $\sigma = \frac{1}{2}$.
- $|\chi(s)|$ non-monotonic passes through three intersection points centered on $\left(\frac{1}{2}, 1\right)$.
- $\zeta(s) = \zeta(1-s)$ for three values of σ centered on $\sigma = \frac{1}{2}$, giving three possibilities for $|\zeta(s)| = 0$.

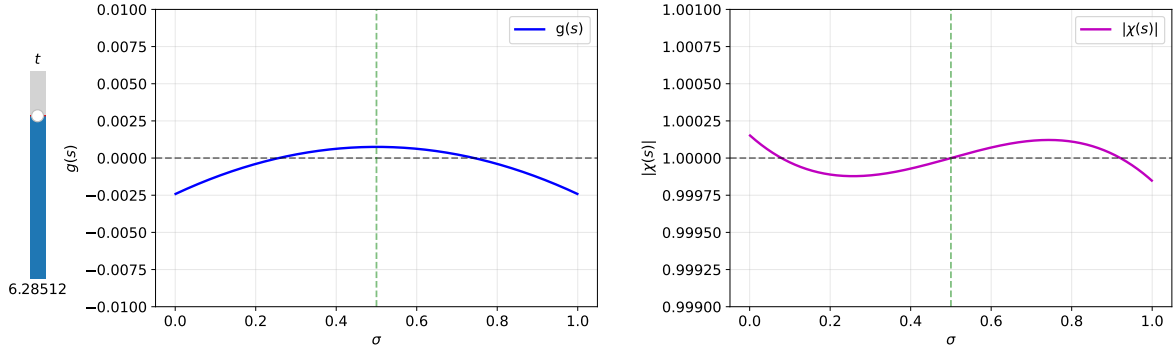


Figure 11: For $t = 6.28512$, the sign of $g(s)$ varies with σ . $|\chi(s)|$ is not monotonic and passes through three points of the line $|\chi(s)| = 1$.

Regime 3: $t \geq t_3 \approx 6.28983$ (Figure 12)

- $g(s) \leq 0$ for all $\sigma \in [0, 1]$; its maximum is global and negative for $t > t_3$
- The asymptotic study shows that $g''(s) \rightarrow 0^-$ when $t \rightarrow +\infty$, confirming the persistence of the concavity of $g(s)$.
- The asymptotic study of $g(s)$ for $t \geq 10$ in the critical strip shows that:

$$g(s) < 0 \quad \forall \sigma \in [0, 1], \quad t \geq 10$$

- The function $|\chi(s)|$ is strictly decreasing in σ .
- $|\chi(s)|$ intersects the line $|\chi(s)| = 1$ only at $\sigma = \frac{1}{2}$.
- On the critical axis $\sigma = \frac{1}{2}$, $|\zeta(s)|$ may potentially vanish.
- If we decrease the value of t , we return to a situation similar to the previous intermediate case (Figure 11), where $|\chi(s)|$ non-monotonic passes through three intersection points centered at $(\frac{1}{2}, 1)$.

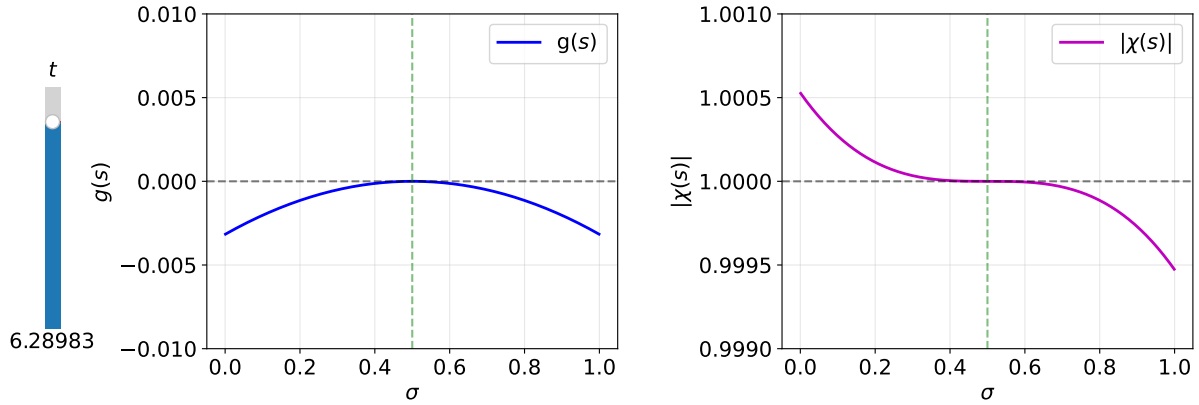


Figure 12: $t_3 \approx 6.28983$ limit from which $|\chi(s)|$ goes from three points to one point on the line $|\chi(s)| = 1$.

4.4.7 Summary

t	Interval	$g(s)$	$ \chi(s) $	Intersections with $ \chi(s) = 1$
$t < t_1$	$t < 6.26985$	> 0	Increasing	1 point $(\frac{1}{2}, 1)$
$t_2 \leq t < t_3$	$6.28318 \leq t < 6.28983$	Variable	Non-monotonic	3 points centered at $(\frac{1}{2}, 1)$
$t \geq t_3$	$t \geq 6.28983$	< 0	Decreasing	1 point $(\frac{1}{2}, 1)$

Table 1: Critical values t_1, t_2, t_3 and their impact on $g(s)$ and $|\chi(s)|$.

Interval	It is possible that $\zeta(s) = 0$ if and only if
$0 < t < t_1$	$\sigma = \frac{1}{2}$
$t_2 \leq t < t_3$	$\sigma = \lambda, \sigma = \frac{1}{2}, \sigma = 1 - \lambda, (0 < \lambda < \frac{1}{2})$. (These values will be rejected in 6.2)
$t_3 \leq t < \infty$	$\sigma = \frac{1}{2}$

Table 2: Values of σ for which $\zeta(s)$ may potentially vanish in the critical strip $\mathcal{D}^+$.

4.4.8 Establishment and Solution of Equations to Calculate the Exact Values of t_1, t_2 and t_3

The precise identification of the thresholds t_1, t_2 and t_3 is essential to characterize the behavioral transitions of $g(s)$ and $|\chi(s)|$ near $t \approx 6.28$.

Environment and Numerical Precision

The calculations were performed in Python, using the `mpmath` library, which allows computations in multiprecision arithmetic. The adopted parameters are as follows:

- Numerical precision: $\text{mp.dps} = 50$ (approximately 50 significant digits);
- Common search interval: $t \in [6.2, 6.4]$.

The precision of the calculations is not critical for what follows, as the analysis relies on the determination of behavioral domains rather than the absolute value of the thresholds. A slight overestimation of t_3 and underestimation of t_2 suffice to guarantee that the domain $\mathcal{D}_R$ entirely covers the study area.

Uniqueness of Solutions For $t > t_0 \approx 0.678157$, the function $g(s)$ is concave and reaches its maximum at $\sigma = \frac{1}{2}$. It is therefore:

- strictly increasing on $(0, \frac{1}{2})$;
- strictly decreasing on $(\frac{1}{2}, 1)$.

This strict monotonicity guarantees the uniqueness of solutions for t_1 and t_3 .

Calculation of t_1 The value t_1 marks the transition where $g(s)$, initially strictly positive, vanishes for the first time. The determining equation is:

$$g(0 + it) = 0 \quad (25)$$

Numerical resolution gives: $t_1 \approx 6.269845775920849$

Calculation of t_2 The value t_2 corresponds to the limit where $|\chi(s)|$ ceases to intersect the line $|\chi(s)| = 1$ at three points. We solve:

$$|\chi(0 + it)| = 1 \quad (26)$$

We obtain: $t_2 \approx 6.283185340798241$

Calculation of t_3 The value t_3 marks the transition where the maximum of $g(s)$ at $\sigma = \frac{1}{2}$ vanishes:

$$g\left(\frac{1}{2} + it\right) = 0 \quad (27)$$

The resolution gives: $t_3 \approx 6.289835988836902$

The uniqueness of t_3 follows from the alternative expression of $|\chi(it)|$ given by the equation :

$$|\chi(it)| = \sqrt{\frac{t}{2\pi} \tanh\left(\frac{\pi t}{2}\right)}$$

5 Numerical Validation of Results

Critical Points and Proof Structure The following critical points, t_0, t_1, t_2, t_3 , determined by Brent's method with an accuracy of 10^{-15} , structure the demonstration:

- $t_0 = 0.678156835255537$: transition from $g(s) > 0$ to the concavity region ($g'' < 0$)
- $t_1 = 6.269845775920849, t_2 = 6.283185340798241, t_3 = 6.289835988836902$: control points for the analysis of $|\chi(s)|$ behavior
- Computations were performed on a Lenovo PC with an AMD Ryzen 7 5825U processor (8 cores, 2.0–4.5 GHz), 16 GB RAM, under Windows 11 64-bit, using mpmath from Python 3.13.5 with $\text{mp.dps} = 50$.
- The 6 scripts below are provided with the repository of this document.

5.1 Proof of $g(s) > 0, 0 < t \leq t_0$

Domain and Method

- $t \in [0.001, t_0], \sigma \in [0, 0.5]$
- Sampling: a grid of $678 \times 500 = 339,000$ uniformly distributed points

Results

- Duration: 238.6 s
- Certification: $g(s) > 0$ for all tested points. By symmetry of function g , the verification is valid for all $\sigma \in [0, 1]$
- Minimum: $g_{\min} = 2.014 > 0$
- Validation of critical point t_0

Remark The concavity of $g(s)$ cannot be determined in this domain.

5.2 Verification of Concavity of $g(s)$ for $t \in [t_0, 10]$

Domain and Method

- Domain: $\sigma \in [0, 1], t \in [t_0, 10]$, with $t_0 = 0.678157$
- Uniform grid: $N_t = 1000$ points in t , $N_\sigma = 101$ points in σ (adjusted to include $\sigma = 0.5$)
- Total number of samples: $N_t \times N_\sigma = 101,000$
- Formula used:

$$g''(s) = -\frac{1}{8} \Re \left[\psi^{(2)}\left(\frac{s}{2}\right) + \psi^{(2)}\left(\frac{1-s}{2}\right) \right],$$

with $\psi^{(2)}$ the polygamma function of order 2

Results

- Duration: 237.16 s
- $g''_{\min} = -6.090089 \times 10^0$ reached at $(\sigma, t) \approx (0.57, t_0)$
- $g''_{\max} = -3.791114 \times 10^{-7}$ reached at $(\sigma, t) = (0, t_0)$
- No violation detected: $g''(s) < 0$ over the entire considered domain

Conclusion: The concavity of $g(s)$ is numerically confirmed for $\sigma \in [0, 1]$ and $t \in [t_0, 10]$.

5.3 Negativity of g for $t \in]t_3, 10]$

Method

- Direct study of $g(s)$ for $t \in]t_3, 10]$ and $\sigma \in [0, 1]$
- Uniform grid: $N_t = 10,000, \sigma = \frac{1}{2}$

Results

- Total duration: 19.52 s
- $g(s) < 0$ over the entire considered domain
- Confirmation of t_3

Remark The negativity of $g(s)$ is easier to verify than its concavity. Showing the negativity of $g''(s)$ is more complex.

5.4 Asymptotic Negativity of $g(s)$ for $t \geq 10$

Domain and Method

- $t \in [10, +\infty[, \sigma \in [0, 1]$
- Uniform asymptotic expansion independent of σ :

$$g_{\text{asym2}}(t) = \log(\pi) - \log\left(\frac{t}{2}\right) + \frac{1}{24t^2}$$

- Numerical determination of the bounding constant:

$$C = 7.430531 \times 10^{-7} + 10^{-12}$$

guaranteeing for all $t \geq 10$ and $\sigma \in [0, 1]$:

$$g(s) < g_{\text{asym2}}(t) + C$$

- Numerical verification performed for $t = 10^1, \dots, 10^{20}$ with $\sigma = \frac{1}{2}$

Results

- Duration: 0.08 s
- Maximum observed value:

$$\max_t (g(1/2 + it) - g_{\text{asym2}}(t)) = 7.33 \times 10^{-7} \text{ at } t = 10$$

- For all $t \geq 10$, $g < g_{\text{asym2}}(t) + C < 0$
- Point verification at $\sigma = 0, 0.25, 0.5, 0.75, 1$ showed no significant variation, confirming σ -independence

Conclusion: The uniform bound

$$g(s) < g_{\text{asym2}}(t) + C < 0$$

for all $t \geq 10$ and $\sigma \in [0, 1]$ completes the numerical certification of the negativity of $g(s)$ in the asymptotic region. Thus,

$$g(s) < 0 \quad \forall \sigma \in [0, 1], t \geq 10$$

5.5 Analysis of Non-Monotonicity of $|\chi(s)|$

Domain and Method

- Domain: $t \in [t_1, t_3[= [6.26985, 6.28983[, \sigma \in [0, 1]$
- Sampling: $N_t = 10,000$ values of t
- Method: verification of the criterion $g(0, t) g(0.5, t) < 0$
 - $g(\sigma, t) = \log(\pi) - \frac{1}{2} \left[\Re \psi \left(\frac{1-\sigma}{2} \right) + \Re \psi \left(\frac{\sigma}{2} \right) \right]$
 - g is concave on $\sigma \in [0, 0.5]$
 - If $g(0, t)$ and $g(0.5, t)$ have opposite signs, g admits a unique zero in $\sigma \in (0, 0.5)$

Results

- Total duration: 12.3 s
- Cases verified: 10,000 out of 10,000 (100%)
- Condition $g(0, t) g(0.5, t) < 0$ verified for all $t \in [t_1, t_3[$
- Therefore a unique zero of $g(\sigma, t)$ for each t in $\sigma \in [0, 0.5]$
- By symmetry, two zeros in $\sigma \in [0, 1]$

Interpretation

- $|\chi(\sigma + it)|$ admits:
 - a minimum to the left of the critical point $(\frac{1}{2}, 1)$
 - a maximum to the right of $(\frac{1}{2}, 1)$
- The modulus $|\chi|$ is therefore **non-monotone** over the domain $t \in [t_1, t_3[, \sigma \in [0, 1]$

5.6 Certification of Three Intersection Points of $|\chi(s)|$ with the Line $|\chi(s)| = 1$

Domain and Method

- $t \in [t_2, t_3[$
- Sampling: $N_t = 1000$ values of t , $N_\sigma = 25,000$ values of $\sigma \in [0, 0.5]$
- Analyzed function: $h(\sigma, t) = |\chi(\sigma + it)| - 1$
- Criterion: detection of a sign change of h in $\sigma \in [0, 0.5]$

Results

- Total duration: 3,330.7 s
- Zero crossing detected for 333/1000 t -values (33.3%)
- First t with crossing: $t \approx 6.283192206937$ (t_2 confirmation)
- Last t with crossing: $t \approx 6.289835407982$ (t_3 confirmation)
- The obtained range correctly brackets t_2 and t_3

Interpretation

- The zero crossings of $h(s) = |\chi(s)| - 1$ correspond to intersections of $|\chi(\sigma + it)|$ with 1
- Three-point configuration by symmetry:
 - $\sigma = 0.5$ (fixed central point)
 - $\sigma = a$ with $0 \leq a < 0.5$
 - $\sigma = 1 - a$ (symmetric)
- Conclusion: $|\chi(s)|$ intersects the line $|\chi(s)| = 1$ at three points for $t \in [t_2, t_3[$

6 Interpretation of Results

6.1 Validity of the Riemann Hypothesis for $0 \leq \sigma \leq 1$ and $(0 < t < t_2$ or $t_3 \leq t < \infty)$

From the study of the function $|\chi(s)|$, (section 4), we can already conclude that all curves of the function $|\chi(s)|$ in the interval $0 \leq \sigma \leq 1$, for t such that $0 < t < t_2$ or $t_3 \leq t < \infty$, intersect the line $|\chi(s)| = 1$ at a single point, namely $(\frac{1}{2}, 1)$, in accordance with the equation $|\chi(\frac{1}{2} + it)| = 1$.

First important result : Since $|\chi(s)| \neq 1$ for all $\sigma \in [0, 1]$, except at $\sigma = \frac{1}{2}$, and for all values of t such that $0 < t < t_2$ or $t \geq t_3$, it follows that $\zeta(s)$ has no zeros in this domain (see lemmas 3.1 and 3.2). We deduce that the Riemann Hypothesis is verified in this region of the critical strip.

For $t_2 \leq t \leq t_3$, it is possible to have $|\chi(s)| = 1$ at three points. It remains to verify, in this small band, that the function $\zeta(s)$ has no zeros, neither at $\sigma = \frac{1}{2}$ nor elsewhere for $0 < \sigma < 1$. For this, we will use the argument principle.

6.2 Absence of Zeros of ζ in the Domain $\{0 \leq \sigma \leq 1, t_2 \leq t < t_3\}$

Recall of the argument principle [5, p. 259, th. 4.4.1].

Let f be a meromorphic function in a simply connected domain $D \subset \mathbb{C}$, and holomorphic on a simple closed contour $\gamma \subset D$, with no zeros or poles on γ . Then :

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = N - P \quad (28)$$

where :

- N = number of zeros of f inside γ (counted with multiplicity),
- P = number of poles of f inside γ (counted with multiplicity).

Geometric interpretation : the integral counts the winding number of $f(\gamma)$ around the origin. If there are no poles, each counterclockwise revolution corresponds to one zero.

We applied the argument principle on the contour of the small rectangle formed by the line segments in figure 13.

Definition of the integration contour.

Given that we cannot know the exact values of t_2 and t_3 , we round these two values to three decimal places, overestimating t_3 and underestimating t_2 , according to the following equations :

$$t'_3 = t_3 \text{ rounded to 3 decimals} + 0.001 = 6.290 + 0.001 = 6.291$$

$$t'_2 = t_2 \text{ rounded to 3 decimals} = 6.283$$

This operation allows complete coverage of the domain of interest D_R along the contour γ .

The rectangle $ABCD$ is then defined by :

$$\begin{aligned} AB &: A(0, 6.283) \rightarrow B(1, 6.283) \\ BC &: B(1, 6.283) \rightarrow C(1, 6.291) \\ CD &: C(1, 6.291) \rightarrow D(0, 6.291) \\ DA &: D(0, 6.291) \rightarrow A(0, 6.283) \end{aligned}$$

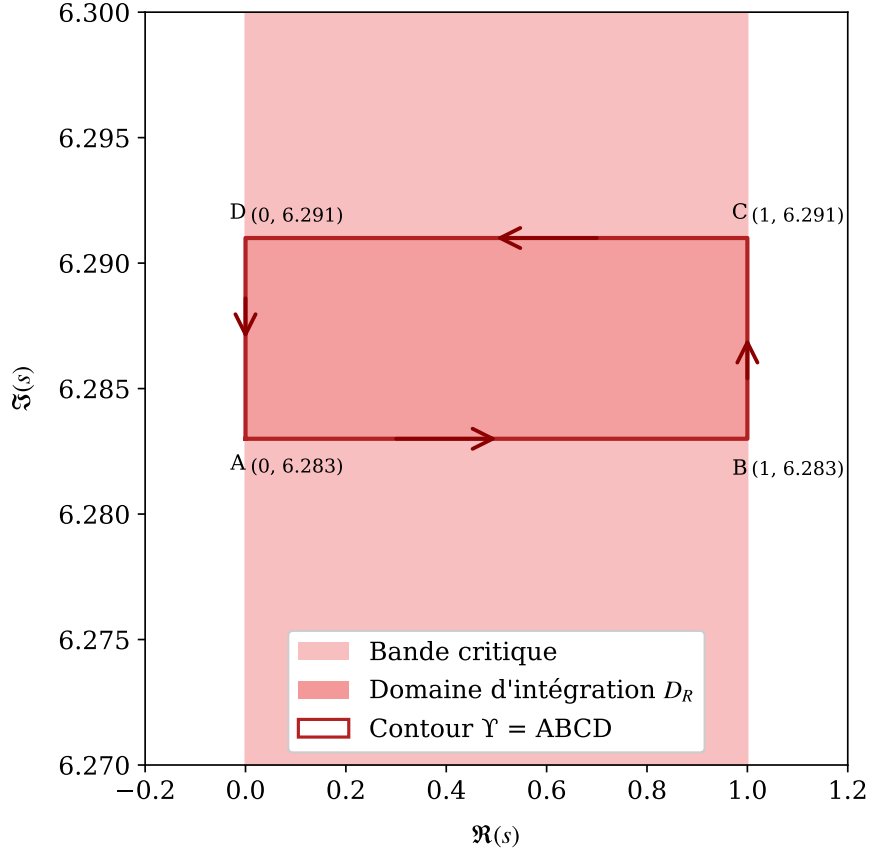


Figure 13: Rectangle defining the integration domain for the application of the argument principle. $A(0, 6.283)$, $B(1, 6.283)$, $C(1, 6.291)$, $D(0, 6.291)$.

Numerical computation: absence of zeros of $\zeta(s)$ in the rectangle $[0, 1] \times [6.283, 6.291]$

The application of the argument theorem was carried out along the rectangular contour

$$\gamma : A(0, 6.283) \rightarrow B(1, 6.283) \rightarrow C(1, 6.291) \rightarrow D(0, 6.291) \rightarrow A(0, 6.283),$$

delimiting the domain

$$D_R = \{ 0 \leq \Re(s) \leq 1, 6.283 \leq \Im(s) \leq 6.291 \}.$$

The computations were performed using the Flint/Arb library (controlled-precision “ball arithmetic”), with the following parameters:

$$\text{mp.dps} = 200, \quad \text{prec_bits} = 1000, \quad N_t = 1000, \quad N_\sigma = 125\,000, \quad \Delta\sigma \approx \Delta t.$$

Vertical sides. On the segments $DA : \Re(s) = 0$ and $BC : \Re(s) = 1$, $\zeta(s)$ does not vanish. Non-vanishing on $\Re(s) = 1$ follows from Hadamard–de la Vallée Poussin [6]; the functional equation then implies the absence of zeros on $\Re(s) = 0$.

Horizontal sides. The minimal moduli of $\zeta(s)$ obtained numerically are:

$$|\zeta(0.5 + i 6.283)| \approx 0.955984159877, \quad |\zeta(0.5 + i 6.291)| \approx 0.957499992414,$$

which rules out any zero of $\zeta(s)$ on the horizontal segments AB and CD .

Argument variation. The total variation of the argument of $\zeta(s)$ along the contour Γ , computed with Flint/Arb, is:

$$\Delta \arg_{\partial D_R} \zeta(s) = [-0.3952306848 \pm 9.3 \times 10^{-11}] \text{ rad.}$$

The corresponding winding number is therefore:

$$\frac{1}{2\pi} \Delta \arg_{\partial D_R} \zeta(s) = [-0.0629029171 \pm 5 \times 10^{-13}] = 0,$$

Conclusion. No zero was detected along the contour, and non-vanishing on the vertical sides is analytically guaranteed. Hence, by the argument theorem (equation (??)), we conclude that:

$$\boxed{\zeta(s) \neq 0 \quad \text{for all } s \in D_R = \{0 \leq \Re(s) \leq 1, 6.283 \leq \Im(s) \leq 6.291\}.$$

This result confirms the absence of any non-trivial zero of $\zeta(s)$ in the domain D_R under consideration.

6.3 Absence of Zeros on the Segment $0 < \sigma < 1, t = 0$

Recall of the well-known proof using Dirichlet's η function. We start from the function :

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \quad (29)$$

For $0 < \sigma < 1$, this alternating series converges and has a positive value according to Leibniz's criterion. Using the relation connecting Dirichlet's η function to Riemann's ζ function [3, p. 16, Eq. (2.2.1)], we can write :

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}. \quad (30)$$

On the interval $0 < \sigma < 1$, the factor $(1 - 2^{1-s})$ is strictly negative, and $\eta(s) > 0$. Consequently, the function $\zeta(s)$ has no zeros on this interval.

6.4 Absence of Zeros on the Vertical Lines $\sigma = 0$ and $\sigma = 1$

The absence of zeros of the function ζ on the line $\sigma = 1$ (excluding the pole $s = 1$) was demonstrated by Hadamard and de la Vallée Poussin in 1896 [6, pp. 204-205]. By symmetry (see section 1.3.2), we deduce that ζ also does not vanish on the line $\sigma = 0$. At the point $s = 0$, we have $\zeta(0) = -\frac{1}{12} \neq 0$.

7 Conclusion

- The function $|\chi(s)| \neq 1$ if $0 \leq \sigma \leq 1$ and $(0 < t < t_2 \text{ or } t_3 < t < \infty)$, which implies that $|\zeta(s)|$ has no zeros outside the critical line $\sigma = \frac{1}{2}$ (see subsection 6.1).
- For $0 \leq \sigma \leq 1$ and $6.283 \leq t \leq 6.291$, $\zeta(s)$ has no zeros (computed using the argument principle) (see section 6.2).
- The two previous results can be extended to $t < 0$ using the symmetry property of $\zeta(s)$ with respect to the σ -axis, by virtue of the conjugation of $\zeta(s)$, see equation (2).
- On the interval $0 < \sigma < 1, t = 0$, $\zeta(s)$ has no zeros (see section 6.3).

- The absence of zeros on the vertical lines $\sigma = 0$ and $\sigma = 1$ was demonstrated in subsection 6.4.

These conclusions are illustrated by figure 14.

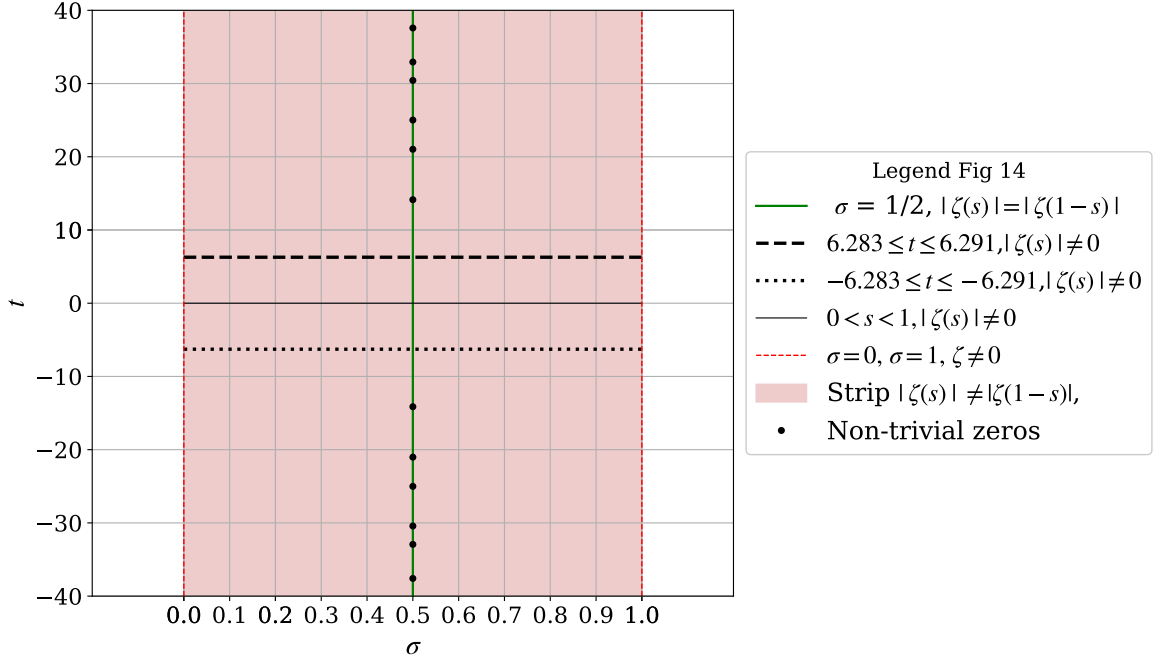


Figure 14: Illustration of the absence of zeros of $\zeta(s)$ in the critical strip, except on the critical line.

Final Result

In the critical strip, the Riemann function $\zeta(s)$ can only have zeros on the critical line $\sigma = \frac{1}{2}$. Consequently, all roots $s_n, n \in \mathbb{N}$, and their complex conjugates $\overline{s_n}$ of the equation $\zeta(s) = 0$ satisfy $\Re(s_n) = \frac{1}{2}$.

The Riemann Hypothesis is valid.

8 Declaration on the Use of Artificial Intelligence

This research was conducted independently, without external funding. The author declares having used the conversational assistants ChatGPT and DeepSeek exclusively for :

- Optimization and debugging of Python scripts for numerical computation and figure generation
- Linguistic reformulation and stylistic correction to improve text clarity
- Technical assistance for equation formatting in $\text{\LaTeX}$

The author assumes full responsibility for the content, rigor of calculations, accuracy of results, and mathematical arguments presented in this work.

References

- [1] Bernhard Riemann and David R. Wilkins. *On the Number of Prime Numbers less than a Given Quantity*. Clay Mathematics Institute / Trinity College Dublin, Dublin, Ireland, 1998. Translated from the German original.
- [2] H. M. Edwards. *Riemann's Zeta Function*. Dover Publications, New York, 1974.
- [3] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function*. Oxford University Press, Oxford, 2nd edition, 1986.
- [4] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. U.S. Government Printing Office, Washington, D.C., 1972.
- [5] Mark J. Ablowitz and Athanassios S. Fokas. *Complex Variables: Introduction and Applications*. Cambridge University Press, Cambridge, 2003.
- [6] Jacques Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bulletin de la Société Mathématique de France*, 24:199–220, 1896.

Appendices

The Python programs presented below are provided for informational purposes only. When copying and pasting from the PDF, indentation may be lost, and some characters (' and ") require manual correction for the programs to run. To avoid these inconveniences, the source files in .py format are also provided in repository of this publication.

Appendix 1: $|\chi(s)|$ function in the critical strip, t variable from 0.1 to 200

```
#-----
# Interactive program showing the evolution of  $|\chi(s)|$ ,  $s=\sigma+it$ 
# in the critical strip  $0 < \sigma < 1$ , for  $t$  variable from 0.1 to 200.
# -----
# -----
# Imports and Library
#-----
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt
from matplotlib.widgets import Slider, Button
import matplotlib as mpl

# Font size configuration
mpl.rcParams.update({
    "font.size": 15,          # default text size
    "axes.labelsize": 15,     # axis labels
    "axes.titlesize": 15,     # titre
    "xtick.labelsize": 15,    # ticks en x
    "ytick.labelsize": 15,    # ticks en y
    "legend.fontsize": 15,    # légende
})
#  $|\chi(s)|$  function
def chi(sigma, t):
    s = sigma + 1j * t
    return abs(np.pi**(s - 0.5) * mp.gamma((1 - s) / 2) / mp.gamma(s / 2))

# Sigma axis
sigma_vals = np.arange(-0.1, 1.1, 0.01)
t0 = 100
chi_vals = [float(chi(sigma, t0)) for sigma in sigma_vals]

# Plot settings
fig, ax = plt.subplots(figsize=(10, 5))

# Choice of dimensions of the inner rectangle
ax.set_position([0.10, 0.16, 0.89, 0.82]) # left, bottom, width, height (0..1)

(line,) = ax.plot(sigma_vals, chi_vals, label=f"t = {t0}", color="blue")
ax.set_xlabel(r" $\sigma$ ")
ax.set_ylabel(r" $|\chi(\sigma+it)|$ ")
ax.axhline(1, color="k", linestyle="--", label=r" $|\chi(s)| = 1$ ")
```

```

ax.axvline(0.5, color="r", linestyle=":", label=r"$\sigma=0.5$")
ax.axvspan(0, 1, color="lightgrey", alpha=1, label="Critical stripe")
ax.set_xlim(-0.1, 1.1)
ax.set_ylim(0, 5)
ax.legend()

# Slider t: place in a very thin area under the graph
ax_slider = fig.add_axes([0.15, 0.06, 0.7, 0.03]) # x, y, width, height
slider = Slider(ax_slider, "t", 0.1, 200, valinit=t0, valstep=0.1)

def update(val):
    t = slider.val
    updated_vals = [float(chi(sigma, t)) for sigma in sigma_vals]
    line.set_ydata(updated_vals)
    line.set_label(f"t = {t:.1f}")
    ax.legend()
    fig.canvas.draw_idle()

slider.on_changed(update)

# Auto-Run Button
ax_button = fig.add_axes([0.82, 0.01, 0.10, 0.04])
button = Button(ax_button, "Auto-Run")

animating = [False]
def auto_run(event):
    animating[0] = not animating[0]
    button.label.set_text("Stop" if animating[0] else "Auto-Run")
    if animating[0]:
        run_auto()

def run_auto():
    if not animating[0]:
        return
    current_t = slider.val
    new_t = current_t + 0.25
    if new_t > 200:
        new_t = 0.1
    slider.set_val(new_t)

    try:
        fig.canvas.manager.window.after(50, run_auto)
    except Exception:
        # Simple fallback: restart via timer
        timer = fig.canvas.new_timer(interval=50)
        timer.add_callback(run_auto)
        timer.start()

button.on_clicked(auto_run)
plt.savefig("image4.pdf")
plt.show()

```

Appendix 2: Simultaneous dynamic evolution of $g(s)$ and $\chi(s)$

```
#-----
# This interactive programme shows the simultaneous evolution of fonctions g(s)
# and |chi(s)| as a function of t in the critical strip.
#-----
# Import and library
#-----
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import gamma, digamma
from matplotlib.widgets import Slider
import matplotlib as mpl
# -----
# Font size configuration
mpl.rcParams.update({
    "font.size": 15,          # default text size
    "axes.labelsize": 15,     # axis labels
    "axes.titlesize": 15,     # title
    "xtick.labelsize": 15,    # ticks en x
    "ytick.labelsize": 15,    # ticks en y
    "legend.fontsize": 15,    # legend
})

# Parameters
# -----
t_min, t_max = 6.23, 6.3
t_init = (t_min + t_max) / 2 # initial slider value (here put your initial value instead of
sigma = np.linspace(0.001, 0.999, 2000)

# -----
# Functions
# -----
def F(s):
    """|Chi(s)|"""
    return abs(np.pi**(s - 0.5) * gamma((1 - s) / 2) / gamma(s / 2))

def g(s):
    """g(s)"""
    return (np.log(np.pi)
            - 0.5 * digamma((1 - s) / 2)
            - 0.5 * digamma(s / 2)).real

# -----
# Figure preparation
# -----
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 4))
plt.subplots_adjust(left=0.18, right=0.995, top=0.98, bottom=0.15, wspace=0.40)
```

```

# Initial value of s
s = sigma + 1j * t_init

# Plot g(s)
line1, = ax1.plot(sigma, g(s), 'b-', lw=2, label=r'$\mathrm{g}(s)$')
ax1.axhline(0, color='k', linestyle='--', alpha=0.5)
ax1.axvline(0.5, color='g', linestyle='--', alpha=0.5)
ax1.set_ylim(-0.01, 0.01)
ax1.set_xlabel(r'$\sigma$')
ax1.set_ylabel(r'$g(s)$')
ax1.grid(True, alpha=0.5)
ax1.legend()

# Plot Chi(s)
line2, = ax2.plot(sigma, F(s), 'm-', lw=2, label=r'$\chi(s)$')
ax2.axhline(1, color='k', linestyle='--', alpha=0.5)
ax2.axvline(0.5, color='g', linestyle='--', alpha=0.5)
ax2.set_ylim(0.999, 1.001)
ax2.set_xlabel(r'$\sigma$')
ax2.set_ylabel(r'$\chi(s)$')
ax2.grid(True, alpha=0.5)
ax2.legend()

# -----
# Slider (for t)
# -----
ax_slider = fig.add_axes([0.04, 0.2, 0.025, 0.6]) # slider position
slider = Slider(
    ax=ax_slider,
    label='$t$',
    valmin=t_min,
    valmax=t_max,
    valinit=t_init,
    orientation='vertical'
)

# -----
# Update function
# -----
def update(val):
    s = sigma + 1j * val
    line1.set_ydata(g(s)) # fixed: call g(s), not G(s)
    line2.set_ydata(F(s))
    fig.canvas.draw_idle()

slider.on_changed(update)
plt.savefig("image8.pdf") # figure from 8 to 12
plt.show()

```

